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Fixed point theorems in complex valued b-metric spaces

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Abstract. In this paper, we have proved common fixed point theorems using Hardy and Rogers type contraction condition in complex-valued b-metric spaces The results of the paper extend the results proved in S. Ali [1].

1. Introduction

In 1922, Banach first proved a fixed point theorem in a complete metric space. This theorem is known as Banach's fixed point theorem. After the work of Banach, many researchers ([6, 11, 12, 15], etc.) have proved several fixed point theorems in many branches of mathematics. The notion of complex-valued metric space was introduced by Azam et al. [2]. Rao et al. [14] extended the notion of complex-valued metric space to complex-valued b-metric space. Dubey et al. [7], Berrah et al. [3], Dubey and Tripathi [8], Ali [1], Sitthikul and Saejung [19], Singh et al. [18], Rouzkard and Imdad [16], Bhardwaj and Wadhwa [4], Hamaizia and Murthy [9], Saluja [17], Bouhadjera [5] have proved several fixed point theorems in complex valued metric spaces and complex-valued b-metric spaces using different conditions on the operators.

It is further observed that Hardy and Rogers [10] have extended Banach fixed point theorem in complete metric spaces. Hardy and Rogers' notions have also been generalized by various researchers. Recently, Mukheimer [13] has proved a uniqueness common fixed point in complete complex valued b-metric spaces.

In this paper we have proved some common fixed theorems using Hardy and Rogers type contraction mappings. Our theorems have generalized the available results in [1].

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2. PRELIMINARIES

With the usual notation $\mathbb{N}, \mathbb{R}, \mathbb{C}$, let $z_1, z_2 \in \mathbb{C}$, we define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Rez_1 \leq Rez_2 \text{ and } Imz_1 \leq Imz_2.$$

Thus we can say, $z_1 \precsim z_2$ if one of the following holds:

(i) $Rez_1 = Rez_2$ and $Imz_1 = Imz_2$,
(ii) $Rez_1 = Rez_2$ and $Imz_1 < Imz_2$,
(iii) $Rez_1 < Rez_2$ and $Imz_1 = Imz_2$,
(iv) $Rez_1 < Rez_2$ and $Imz_1 < Imz_2$.

We write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and any one of (ii), (iii) and (iv) is satisfied. If only the condition (iv) hold, then we write $z_1 \prec z_2$.

It is clear that

(i) $z_1 \precsim z_2$ and $z_2 \prec z_3$ implies $z_1 \prec z_3$,
(ii) $a, b \in \mathbb{R}$ and $a < b$, then $az \precsim bz$, for all $z \in \mathbb{C}$,
(iii) $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

3. DEFINITIONS

Azam et al. [2] defined the complex valued metric space as follows.

Definition 1. A complex valued metric on a non-empty set X is a mapping $d : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$, the following conditions holds:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

Then the pair (X, d) is called a complex valued metric space.

Definition 2 ([14]). A complex valued metric on a non-empty set X is a mapping $d : X \times X \to \mathbb{C}$, such that for all $x, y, z \in X$, the following conditions holds:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) there exists a real number $s \geq 1$ such that $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

Then the pair (X, d) is called a complex valued b-metric space with coefficient $s \geq 1$.

Example 1 ([14]). Let $X = [0, 1]$. Define the mapping $d : X \times X \to C$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 3 ([14]). Let (X, d) be a complex valued b-metric space and $A \subset X$. We recall the following definitions:

(i) $a \in A$ is called an interior point of the set A whenever there is $0 \prec r \in \mathbb{C}$, such that
$$N(a,r) \subset A,$$
where $N(a,r) = \{x \in X : d(a,y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,
$$N(x,r) \cap (A \setminus \{x\}) \neq \phi.$$

(iii) A subset $A \subset X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subset X$ is called closed whenever each limit point of A belongs to A.

The collection $F = \{N(x,r) : x \in X.0 \prec r\}$ is a sub-basis for a topology on X. The topology is denoted by τ. It is to be noted that this topology τ is Hausdorff topology.

Definition 4 ([14])**.** Let (X,d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X and $x \in X$. We call

(i) the sequence $\{x_n\}$ converges to x if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \prec c$. We write this as $\lim_{n\to\infty} x_n = x$ or, $x_n \to x$ as $n \to \infty$;

(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$ and $m \in \mathbb{N}, d(x_n,x_m) \prec c$;

(iii) The metric space (X,d) is a complete complex valued b-metric space if every Cauchy sequence is convergent in X.

Azam et al. [2] established the following lemmas.

Lemma 1. *Let (X,d) be a complex valued b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2. *Let (X,d) be a complex valued b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n,m \to \infty$.*

4. Main Results

Our main results are as follows.

Theorem 1. *Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be self-maps satisfying the following condition:*

$$d(fx,gy) \preceq \alpha d(x,y) + \beta \max\Big\{d(x,y), \frac{d(x,fx)d(y,gy)}{1+d(fx,gy)}\Big\} + \gamma \min\big\{d(x,gy), d(y,fx)\big\}, \tag{1}$$

where $\alpha+\beta+s\gamma<1$, $\alpha,\beta,\gamma\geq 0$. *Then* f *and* g *have unique common fixed point in* X.

Proof. Let $x_0\in X$ be an arbitrary. We construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1}=fx_{2n}, \qquad x_{2n+2}=gx_{2n+1}.$$

Now,

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(fx_{2n},gx_{2n+1})\\
&\precsim \alpha d(x_{2n},x_{2n+1})\\
&\quad+\beta\max\Big\{d(x_{2n},x_{2n+1}),\frac{d(x_{2n},fx_{2n})d(x_{2n+1},gx_{2n+1})}{1+d(fx_{2n},gx_{2n+1})}\Big\}\\
&\quad+\gamma\min\big\{d(x_{2n},gx_{2n+1}),d(x_{2n+1},fx_{2n})\big\}\\
&= \alpha d(x_{2n},x_{2n+1})\\
&\quad+\beta\max\Big\{d(x_{2n},x_{2n+1}),\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+1},x_{2n+2})}\Big\}\\
&\quad+\gamma\min\big\{d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1})\big\}\\
&= \alpha d(x_{2n},x_{2n+1})+\beta d(x_{2n},x_{2n+1})+\gamma.0\\
&= (\alpha+\beta)d(x_{2n},x_{2n+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|d(x_{2n+1},x_{2n+2})| &\leq |(\alpha+\beta)d(x_{2n},x_{2n+1})|\\
&\leq (\alpha+\beta)^2|d(x_{2n-1},x_{2n})|\\
&\leq \cdots\\
&\leq (\alpha+\beta)^{2n+1}|d(x_0,x_1)|.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}|d(x_{2n+1},x_{2n+2})|=0 \text{ [since } \alpha+\beta<1].$$

Again let, $n,m\in\mathbb{N},n\geq m$. Then,

$$\begin{aligned}
&d(x_{n+1},x_{m+1})=d(fx_n,gx_m)\\
&\precsim \alpha d(x_n,x_m)+\beta\max\Big\{d(x_n,x_m),\frac{d(x_n,fx_n)d(x_m,gx_m)}{1+d(fx_n,gx_m)}\Big\}\\
&\quad+\gamma\min\{d(x_n,gx_m),d(x_m,fx_n)\}\\
&=\alpha d(x_n,x_m)+\beta\max\Big\{d(x_n,x_m),\frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_{n+1},x_{m+1})}\Big\}\\
&\quad+\gamma\min\{d(x_n,x_{m+1}),d(x_m,x_{n+1})\}\\
&\precsim \alpha d(x_n,x_m)+\beta\max\Big\{d(x_n,x_m),\frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_{n+1},x_{m+1})}\Big\}\\
&\quad+\gamma\min\big\{s[d(x_n,x_m)+d(x_m,x_{m+1})],s[d(x_m,x_n)+d(x_n,x_{n+1})]\big\}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}|d(x_{n+1},x_{m+1})| \leq (\alpha+\beta)\lim_{n\to\infty}|d(x_n,x_m)| + \lim_{n\to\infty}\ \gamma\ s\ |d(x_n,x_m)|$$

implies

$$\lim_{n\to\infty}|d(x_n,x_m)| \leq (\alpha+\beta+s\gamma)\lim_{n\to\infty}|d(x_n,x_m)|$$

implies

$$\lim_{n\to\infty}|d(x_n,x_m)| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is a complete complex valued b-metric space, there exists an $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Therefore,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n-1} = u = \lim_{n\to\infty} gx_n.$$

Now,

$$\begin{aligned} d(fu,u) &\precsim s[d(fu,x_{n+1}) + d(x_{n+1},u)] \\ &= s\ d(fu,gx_n) + sd(x_{n+1},u) \\ &\precsim s[\alpha d(u,x_n) + \beta\max\Big\{d(u,x_n), \frac{d(u,fu)d(x_n,gx_n)}{1+d(fu,gx_n)}\Big\} \\ &\quad + \gamma\min\{d(u,gx_n), d(x_n,fu)\}] + sd(x_{n+1},u) \\ &= s[\alpha d(u,x_n) + \beta\max\Big\{d(u,x_n), \frac{d(u,fu)d(x_n,x_{n+1})}{1+d(fu,x_{n+1})}\Big\} \\ &\quad + \gamma\min\{d(u,x_{n+1}), d(x_n,fu)\}] + sd(x_{n+1},u), \end{aligned}$$

which implies, $\lim_{n\to\infty}|d(fu,u)| \to 0$.

Thus, $|d(fu,u)| = 0$ implies $fu = u$. So u is a fixed point of f.

Again,

$$\begin{aligned} d(u,gu) &\precsim s[d(u,x_{x+1}) + d(x_{n+1},gu)] \\ &= sd(u,x_{n+1}) + sd(fx_n,gu) \\ &\precsim sd(u,x_{n+1}) + s[\alpha d(x_n,u) + \beta\max\Big\{d(x_n,u), \frac{d(x_n,fx_n)d(u,gu)}{1+d(fx_n,gu)}\Big\} \\ &\quad + \gamma\min\{d(x_n,gu), d(u,fx_n)\}] \\ &= sd(u,x_{n+1}) + s[\alpha d(x_n,u) + \beta\max\Big\{d(x_n,u), \frac{d(x_n,x_{n+1})d(u,gu)}{1+d(x_{n+1},gu)}\Big\} \\ &\quad + \gamma\min\{d(x_n,gu), d(u,x_{n+1})\}], \end{aligned}$$

which implies

$$\lim_{n\to\infty}|d(u,gu)| = 0,$$

implies

$$gu = u.$$

Thus u is a common fixed point of f and g.

Let, v be another common fixed point f and g. Then,

$$\begin{aligned} d(u,v) &= d(fu,gv) \\ &\preceq \alpha d(u,v) + \beta \max\Big\{d(u,v), \frac{d(u,fu)d(v,gv)}{1+d(fu,gv)}\Big\} \\ &\quad + \gamma \min\{d(u,gv), d(v,fu)\} \\ &= \alpha d(u,v) + \beta \max\Big\{d(u,v), \frac{d(u,u)d(v,v)}{1+d(u,v)}\Big\} \\ &\quad + \gamma \min\{d(u,v), d(v,u)\} \\ &= (\alpha+\beta+\gamma)d(u,v) \end{aligned}$$

implies

$$(1-\alpha-\beta-\gamma)|d(u,v)| = 0,$$

implies

$$|d(u,v)| = 0,$$

i.e., $u = v$.

Thus f and g have unique common fixed point in X. □

Corollary 1. *Let (X,d) be a complete complex valued b-metric space with coefficient $s \leq 1$ and $f,g : X \to X$ be self-maps satisfying the following condition:*

$$d(fx,gy) \preceq \beta \max\Big\{d(x,y), \frac{d(x,fx)d(y,gy)}{1+d(fx,gy)}\Big\},$$

where $0 \leq \beta < 1$. Then f and g have unique common fixed point in X.

*This result is **Theorem 1** of S. Ali [1].*

Corollary 2. *Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$\begin{aligned} d(fx,fy) \preceq \alpha d(x,y) &+ \beta \max\Big\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\Big\} \\ &+ \gamma \min\{d(x,fy), d(y,fx)\}, \end{aligned}$$

where $\alpha+\beta+s\gamma < 1$, $\alpha,\beta,\gamma \geq 0$. Then f have unique fixed point in X.

Corollary 3. *Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$d(fx,fy) \preceq \alpha d(x,y) + \beta \max\Big\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\Big\},$$

where $\alpha+\beta < 1$, $\alpha,\beta \geq 0$.

Corollary 4. *Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha d(x, y),$$

where $0 \leq \alpha < 1$. Then f have a unique fixed point in X.

This result is ***Banach Theorem*** *in complete complex valued b-metric space.*

Example 2. Let $X = \mathbb{C}$ and $d : X \times X \to \mathbb{C}$ be defined by $d(x, y) = i|x-y|^2$. Also let $f, g : X \to X$ be given by $fx = \frac{x}{2}, gx = \frac{x}{3}$.

Then clearly

- (i) $0 \precsim i|x-y|^2 = d(x, y)$ and $d(x, y) = i|x-y|^2 = 0$ if and only if $|x-y| = 0$ i.e., $x = y$.
- (ii) $d(x, y) = d(y, x)$.
- (iii) $d(x, y) = i|x - y^2| = i|(x-z)+(z-y)|^2 \precsim i\{|x-z|^2 + |z-y|^2 + 2|x-z||z-y|\} \precsim 2i[|x-z|^2 + |z-y|^2] = 2[d(x-z) + d(z-y)]$.

Thus (X, d) is a complex valued b-metric space with coefficient $s = 2$.

Now consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n+1}$ for $i = 0, 1, 2, \ldots$, with initial approximation $x_0 = 1$ given by $x_n = fx_{n-1}$ and $x_{n+1} = gx_n$.

Again,

$$\begin{aligned}
d(x, y) &= i|x-y|^2, \\
d(fx, gy) &= i|fx - gy|^2 = i|\tfrac{x}{2} - \tfrac{y}{3}|^2, \\
d(x, fx) &= i|x - fx|^2 = i|x - \tfrac{x}{2}| = i|\tfrac{x}{2}|^2, \\
d(y, fy) &= i|y - gy|^2 = i|y - \tfrac{y}{3}|^2 = i|\tfrac{2y}{3}|^2, \\
d(x, gy) &= i|x - gy|^2 = i|x - \tfrac{y}{3}|^2, \\
d(y, fx) &= i|y - fx|^2 = i|y - \tfrac{x}{2}|^2.
\end{aligned}$$

Since

$$d(x, fx)d(y, fy) = i|\tfrac{x}{2}|^2 i|\tfrac{2y}{3}|^2 = -|\tfrac{xy}{9}|,$$

$$\max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, gy)}\right\} = d(x, y) = i|x-y|^2.$$

Also, $\min\{d(x, gy), d(y, fx)\} \precsim d(x, y)$. Therefore the condition of (1) is satisfied. So by **Theorem 1**, f and g have unique common fixed point $'0 + i0'$.

Theorem 2. *Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx),$$

where each of $\alpha_i \geq 0$ and $\alpha_1 + s\alpha_2 + \alpha_3 + 2s\alpha_4 + s\alpha_5 < 1$. Then f have a unique fixed point in X.

Proof. Let $x_0 \in X$ be an initial point. We construct a sequence $\{x_n\} \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$.

At first we show that $\lim_{n\to\infty} |d(x_n, x_{n+1})| = 0$.

Since,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\
&\precsim \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, fx_{n-1}) + \alpha_3 d(x_n, fx_n) \\
&\quad + \alpha_4 d(x_{n-1}, fx_n) + \alpha_5 d(x_n, fx_{n-1}) \\
&= \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) \\
&\quad + \alpha_4 d(x_{n-1}, x_{n+1}) + \alpha_5 d(x_n, x_n) \\
&\precsim (\alpha_1 + \alpha_2) d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) \\
&\quad + \alpha_4 s[d(x_{n-1}, x_n) + d(x_n, x_{n+1}))] + \alpha_5 .0 \\
&= (\alpha_1 + \alpha_2 + s\alpha_4) d(x_{n-1}, x_n) + (\alpha_3 + s\alpha_4) d(x_n, x_{n+1})
\end{aligned}$$

which implies

$$(1 - \alpha_3 - s\alpha_4) d(x_n, x_{n+1}) \precsim (\alpha_1 + \alpha_2 + s\alpha_4) d(x_{n-1}, x_n)$$

implies

$$\begin{aligned}
d(x_{n-1}, x_n) &\precsim (\frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - \alpha_3 - s\alpha_4}) d(x_{n-1}, x_n) \\
&= kd(x_{n-1}, x_n), \quad \text{where } k = \left(\frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - \alpha_3 - s\alpha_4}\right) \\
&\precsim k^2 d(x_{n-2}, x_{n-1}) \\
&\quad \vdots \\
&\precsim k^n d(x_0, x_1).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} |d(x_n, x_{n+1})| = 0.$$

Now let, $n, m \in \mathbb{N}$ and $n \geq m$. Then

$$\begin{aligned}
d(x_m, x_n) &= d(fx_{m-1}, fx_{n-1}) \\
&\precsim \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, fx_{m-1}) + \alpha_3 d(x_{n-1}, fx_{n-1}) \\
&\quad + \alpha_4 d(x_{m-1}, fx_{n-1}) + \alpha_5 d(x_{n-1}, fx_{m-1}) \\
&= \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, x_m) + \alpha_3 d(x_{n-1}, x_n) + \\
&\quad \alpha_4 d(x_{m-1}, x_n) + \alpha_5 d(x_{n-1}, x_m) \\
&\precsim \alpha_1 d(x_{m-1}, x_{n-1}) + \alpha_2 d(x_{m-1}, x_m) + \alpha_3 d(x_{n-1}, x_n) \\
&\quad + \alpha_4 s[d(x_{m-1}, x_m) + d(x_m, x_n)] \\
&\quad + \alpha_5 s[d(x_{n-1}, x_n) + d(x_n x_m)].
\end{aligned}$$

Taking modulus and limit as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} |d(x_m, x_n)| \leq \alpha_1 \lim_{n\to\infty} |d(x_{m-1}, x_{n-1})| + \alpha_2.0 + \alpha_3.0 \\ + (\alpha_4 s + \alpha_5 s) \lim_{n\to\infty} |d(x_m, x_n)|\end{aligned}$$

implies

$$(1 - \alpha_1 - \alpha_4 s - \alpha_5 s) \lim_{n\to\infty} |d(x_m, x_n)| \leq 0$$

implies

$$\lim_{n\to\infty} |d(x_m, x_n)| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since the space is complete, there exists an $x \in X$ such that $\lim_{n\to\infty} |d(x_n, x)| = 0$. Now we show that x is a fixed point of f.

Again,

$$\begin{aligned}d(fx, x) &\preceq s[d(fx, fx_n) + d(fx_n, x)] \\ &\precsim s[\alpha_1 d(x, x_n) + \alpha_2 d(x, fx) + \alpha_3 d(x_n, fx_n) + \alpha_4 d(x, fx_n) \\ &\quad + \alpha_5 d(x_n, fx) + d(x_{n+1}, x)] \\ &= s[\alpha_1 d(x, x_n) + \alpha_2 d(x, fx) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x, x_{n+1}) \\ &\quad + \alpha_5 d(x_n, fx) + d(x_{n+1}, x)]\end{aligned}$$

implies

$$\begin{aligned}\lim_{n\to\infty} |d(fx, x)| \leq s[\alpha_1.0 + \alpha_2 |d(fx, x)| + \alpha_3.0 + \alpha_4.0 \\ + \alpha_5 \lim_{n\to\infty} |d(x_n, fx)| + 0].\end{aligned} \tag{2}$$

Again,

$$\begin{aligned}d(x_n, fx) &= d(fx_{n-1}, fx) \\ &\preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, fx_{n-1}) + \alpha_3 d(x, fx) \\ &\quad + \alpha_4 d(x_{n-1}, fx) + \alpha_5 d(x, fx_{n-1}) \\ &= \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x, fx) \\ &\quad + \alpha_4 d(x_{n-1}, fx) + \alpha_5 d(x, x_n).\end{aligned} \tag{3}$$

If $d(x_{n-1}, fx) \preceq d(x, fx)$, then from (3) we have

$$d(x_n, fx) \preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + (\alpha_3 + \alpha_4) d(x, fx) + \alpha_5 d(x_n, x).$$

Therefore,

$$\lim_{n\to\infty} |d(x_n, fx)| \leq (\alpha_3 + \alpha_4) |d(x, fx)|.$$

From (2), we get

$$\lim_{n\to\infty} |d(x, fx)| \leq (\alpha_2 + \alpha_3 + \alpha_4) |d(x, fx)|$$

implies $|d(x, fx)| = 0$, i.e., $fx = x$.

Again if $d(x, fx) \preceq d(x_{n-1}, fx)$, then from (3), we get $d(x_n, fx) \preceq \alpha_1 d(x_{n-1}, x) + \alpha_2 d(x_{n-1}, x_n) + (\alpha_3+\alpha_4) d(x_{n-1}, fx) + \alpha_5 d(x_n, x)$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} |d(x_n, fx)| &\leq (\alpha_3+\alpha_4) \lim_{n\to\infty} d(x_{n-1}, fx)\\ &\leq (\alpha_3+\alpha_4)^2 \lim_{n\to\infty} d(x_{n-2}, fx)\\ &\ \ \vdots\\ &\leq (\alpha_3+\alpha_4)^{n-1} \lim_{n\to\infty} d(x_0, fx).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} |d(x_n, fx)| = 0.$$

Thus we get from (2),

$$|d(fx, x)| \leq s\alpha_2 |d(x, fx)|$$

implies

$$(1-\alpha_2 s)|d(fx, x)| \leq 0$$

implies $|d(fx, x)| = 0$, i.e., $fx = x$. Therefore, F have a fixed point.

To show that x is unique let, y be another fixed point of f. Then we get

$$\begin{aligned}d(x, y) &= d(fx, fy)\\ &\preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx)\\ &= \alpha_1 d(x, y) + \alpha_2 d(x, x) + \alpha_3 d(y, y) + \alpha_4 d(x, y) + \alpha_5 d(y, x)\end{aligned}$$

implies

$$(1-\alpha_1-\alpha_4-\alpha_5)|d(x, y)| = 0$$

implies, $x = y$.

Thus f have a unique fixed point in X. ☐

Corollary 5. *Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_1 d(x, y),$$

where each of $0 \leq \alpha_1 < 1 \geq 0$. Then f have a unique fixed point in X.

*This result is **Banach** contraction condition in complex valued b-metric space.*

Corollary 6. *Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_2 [d(x, fx) + d(y, fy)],$$

where each of $\alpha_2 \geq 0$ and $s\alpha_2 = \alpha_3 < \frac{1}{2}$. Then f have a unique fixed point in X.

This result is ***Kannan*** *contraction condition in complex valued b-metric space.*

Corollary 7. *Let* (X, d) *be a complete complex valued b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_4[d(x, fy) + d(y, fx)],$$

where each of $\alpha_4 \geq 0$ *and* $s\alpha_4 = s\alpha_5 < \frac{1}{2}$. *Then* f *have a unique fixed point in* X.

This result is ***Chatterjea*** *contraction condition in complex valued b-metric space.*

Corollary 8. *Let* (X, d) *be a complete complex valued b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be self-map satisfying the following condition:*

$$d(fx, fy) \preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy),$$

where each of $\alpha_i \geq 0$ *and* $\alpha_1 + s\alpha_2 + \alpha_3 < 1$. *Then* f *have a unique fixed point in* X.

This result is ***Reich*** *contraction condition in complex valued b-metric space.*

5. Conclusion

In this article we have extended Hardy and Roger's [10] result in complex-valued *b*-metric spaces. This result has also extended the results of **Kannan**, **Chatterjea**, **Reich**, etc. We have provided an example in support of condition used in our theorems.

Further, the obtained results scope for extension of many results available in the literature in future.

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